Chaotic properties of the truncated elliptical billiard

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Abstract

Chaotic properties of symmetrical two-dimensional stadium-like billiards with elliptical arcs are studied numerically and analytically. For the two-parameter truncated elliptical billiard the existence and linear stability of several lowest-order periodic orbits are investigated in the full parameter space. Poincaré plots are computed and used for evaluation of the degree of chaoticity with the box-counting method. The limit of the fully chaotic behavior is identified with circular arcs. Above this limit, for flattened elliptical arcs, mixed dynamics with numerous stable elliptic islands is present, similarly as in the elliptical stadium billiards. Below this limit the full chaos extends over the whole region of elongated shapes and the existing orbits are either unstable or neutral. This is conspicuously different from the behavior in the elliptical stadium billiards, where the chaotic region is strictly bounded from both sides. To examine the mechanism of this difference, a generalization to a novel three-parameter family of boundary shapes is proposed and suggested for further evaluation.

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I.   INTRODUCTION

Two-dimensional planar billiards are nonlinear systems with rich and interesting dynamical properties. A point particle, moving with constant velocity within a closed boundary and exhibiting specular reflections on the walls, can have regular, mixed or fully chaotic dynamics, in strong dependence on details of the boundary shape. In physics, two-dimensional billiards offer good examples of coexistence of regular, mixed and chaotic dynamics in Hamiltonian systems. This type of behavior, illustrated by the standard map and explained by means of the KAM-theorem, is present in many realistic phenomena, such as planetary systems and various types of coupled oscillators[1]. Chaotic billiards were first introduced by Sinai[2] who considered the defocusing effects of circular scatterers in the two-dimensional Lorentz gas. After the important discovery by Bunimovich[3, 4] that also the focusing circular arcs can lead to a fully chaotic behavior, many investigations were devoted to billiards with circular arcs and, in a smaller extent, to other types of curved boundaries. The systematic mathematical description of chaotic billiards and an extended list of references can be found in the book by Chernov and Markarian[5]. Rigorous investigations were concentrating on methods for producing fully chaotic billiards and on specific properties (Bernoulli, K-property, mixing and hyperbolicity) expressing differences between chaotic systems[5, 6, 7, 8, 9]. Various aspects of billiard dynamics have been extensively examined during last decades[10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. In recent years, properties of classical billiards and their quantum-mechanical counterparts were used to explain and improve performances of devices in microelectronics and nanotechnology, especially of optical microresonators in dielectrical and polymer lasers[20, 21, 22, 23, 24, 25].

We are stressing the fact that notable regions of full chaos have been discovered in billiards with elliptical arcs and piecewise flat boundaries, indicating that such billiards deserve further attention[26, 27, 28, 29]. In our previous work we analyzed several types of billiards with noncircular arcs (parabolic, hyperbolic, elliptical and generalized power-law), exhibiting mixed dynamics[30, 31, 32]. Next we investigated, in the full parameter space[33], the elliptical stadium billiards (ESB), first introduced by Donnay[7]. Here we extend the same type of analysis to the truncated elliptical billiards (TEB), which although similar in appearance, have different dynamical properties. The truncated elliptical billiard (TEB) is defined by a two-parameter planar domain constructed by truncating an ellipse on opposite sides

(Fig. 1). A symmetrical stadium-like shape thus obtained consists of a rectangle with two elliptical arcs added at its opposite ends. The corresponding billiard has been introduced by Del Magno[29] who, investigating a restricted part of the parameter space and applying the mathematical method of invariant cones, determined the region of hyperbolic behavior and presented an estimate of the region where such billiard could be ergodic.

In the present paper we investigate numerically and analitically the truncated elliptical billiard (TEB) in the full parameter space, by using two shape parameters δ and γ. This description of the billiard geometry and dynamics is consistent with our previous analysis of the elliptical stadium billiard (ESB)[30, 33, 34], which is a two-parameter generalization of the Bunimovich stadium billiard[4] and is a special case of the mushroom billiard[35, 36, 37]. It has been confirmed by analysis and numerical computation[7, 26, 27, 28, 33, 34] that this billiard is fully chaotic (ergodic) for a sizeable but strictly limited region in the parameter space, defined by the stable two-bounce horizontal periodic orbit on one and the pantografic orbits on the other side. Our investigations of the ESB and TEB billiards confirm the suggestion by Del Magno[29] that in spite of apparently similar stadium-like shapes, these two billiards have essentially different dynamical properties. In the present paper we describe our analytical and numerical investigation of the truncated elliptical billiard and compare the obtained results with those for the elliptical stadium billiard.

In Section II we define the TEB billiard boundary and describe its geometrical properties. In Section III the existence and stability of selected orbits are discussed and illustrated by Poincaré plots and orbit diagrams. In Section IV the Poincaré sections are used to estimate, by means of the box-counting numerical method, the degree of chaoticity for a given boundary shape. The results are shown in the parameter-space diagram and compared with the same type of diagram for the elliptical stadium billiard. In Section V we briefly discuss the possible generalization of the truncated elliptical billiard providing a transition between two types of the stadium-like elliptical billiards. Finally, in Section VI we summarize the obtained results and propose further investigations.

FIG. 1: Three types of the TEB billiard shape: (a) circular shape, with $\delta = \sqrt{1-\gamma^2}$: $\delta = 0.6$, $\gamma = 0.8$; (b) flattened shape with elliptical arcs and $\delta > \sqrt{1-\gamma^2}$: $\delta = 0.6$, $\gamma = 2.2$ and (c) elongated shape with $\delta < \sqrt{1-\gamma^2}$: $\delta = 0.6$, $\gamma = 0.2$

II. GEOMETRICAL PROPERTIES OF THE TRUNCATED ELLIPTICAL BILLIARD

In our parametrization the truncated elliptical billiard (TEB) is defined in the $x-y$ plane by means of the two parameters δ and γ, satisfying conditions $0 \leq \delta \leq 1$ and $0 < \gamma < \infty$. The billiard boundary is described as

$$
y(x) = \begin{cases} \pm\gamma & \text{, if } 0 \leq |x| < \delta \\[2ex] \pm\gamma\sqrt{\frac{1-x^2}{1-\delta^2}} & \text{, if } \delta \leq |x| \leq 1 \end{cases} \tag{1}
$$

The horizontal diameter is normalized to 2, so that the horizontal semiaxis of the ellipse is 1. The vertical semiaxis of the ellipse is $\gamma/\sqrt{1-\delta^2}$, and the possible height 2γ of the billiard extends from 0 to ∞. The horizontal length of the central rectangle is 2δ.

In special cases, for $\delta = 0$ the shape is a full ellipse, for $\delta = 1$ it is rectangular, for $\delta = \gamma = 1$ it is a square and for $\delta = 0$ and $\gamma = 1$ a full circle. For $\delta = \sqrt{1-\gamma^2}$ one obtains a set of truncated circle billiards, which separates two distinct billiard classes, one for $\delta < \sqrt{1-\gamma^2}$ with elongated elliptical arcs and the other with $\delta > \sqrt{1-\gamma^2}$ and flattened elliptical arcs. Fig. 1 shows three typical shapes of the truncated elliptical billiard with circular, flattened and elongated elliptical arcs.

The coordinates of the focal points are

$$
F\left[\pm\sqrt{\frac{1-\delta^2-\gamma^2}{1-\delta^2}}, 0\right] \tag{2}
$$

for $\delta < \sqrt{1-\gamma^2}$, and

$$
F\left[0, \pm\sqrt{\frac{\gamma^2+\delta^2-1}{1-\delta^2}}\right] \tag{3}
$$

for $\delta > \sqrt{1-\gamma^2}$. They contain the important term $\tau = \gamma^2 + \delta^2 - 1$ which is negative for $\delta < \sqrt{1-\gamma^2}$, positive for $\delta > \sqrt{1-\gamma^2}$, and zero for $\delta = \sqrt{1-\gamma^2}$ (circular arcs). This limit

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FIG. 2: Diagram of the two-dimensional parameter space (γ, δ). Lines denote the limits of existence and stability for certain orbits, as explained in the text.

is shown as the thick circular line in Fig. 2 presenting the structure of the $\gamma - \delta$ parameter space.

For $\delta < |x| \leq 1$ the curvature radius is

$$R = \frac{[(1 - \delta^2 - \gamma^2)(1 - x^2) + \gamma^2]^{3/2}}{\gamma(1 - \delta^2)} \qquad (4)$$

For $0 \leq |x| < \delta$ the boundary is flat and the curvature radius is $R = \infty$, but for $|x| = \delta$ has a discontinuity and drops to

$$R_\delta = \frac{[(1 - \delta^2)^2 + \gamma^2\delta^2]^{3/2}}{\gamma(1 - \delta^2)} \qquad (5)$$

At the endpoints of the horizontal axis of the ellipse $(|x| = 1)$ the curvature radius is

$$R_1 = \frac{\gamma^2}{1 - \delta^2} \qquad (6)$$

which reduces to $R_1 = 1$ for circular arcs. For full ellipses with $\delta = 0$ the curvature radius at $x = 0$ is $R_0 = 1/\gamma$.

As explained in [33], the symbols θ, ϕ and ϕ', respectively, denote the angles which the normal, the incoming path and the outcoming path make with the x-axis. The angle between the incoming (or outcoming) path and the normal to the boundary is $\beta = (\phi' - \phi)/2$. The angle between the tangent to the boundary at the point $T(x, y)$ of impact and the incoming (or outcoming) path, needed in the computation of the orbit stability, is $\alpha = (\pi/2) - \beta$.

The angles θ, ϕ and ϕ' are connected by the relation

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{\tan\phi + \tan\phi'}{1 - \tan\phi\tan\phi'} \qquad (7)$$

The expression (7) is the basis for finding the existence criteria for particular periodic orbits[33]. In our further description we refer to the impact points $T(x, y)$ in the first quadrant, with no loss of generality for the obtained results. In the Poincaré sections the points $P(x, v_x)$ are obtained by plotting the slope of the velocity direction $v_x = \cos\phi$ versus the x-coordinate of the intersection point with the x-axis, as explained in [30, 31, 32, 33]. The Poincaré diagrams obtained in this way are area preserving.

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FIG. 3: Poincaré plots for $\delta = 0.19$ and various γ.

As described in [13, 33], the stability of a periodic orbit is assured if the absolute value of the trace of the stability matrix M is smaller than 2, thus if

$$-2 < \mathrm{Tr}M < 2 \tag{8}$$

Such orbits are elliptic, and those with $|\mathrm{Tr}M| = 2$ are neutral (parabolic). The stability matrix of the closed orbit of period N can be written as $M = M_{12}M_{23}..M_{N1}$, where the 2×2 matrix M_{ik} for two subsequent impact points T_i and T_k, connected by a rectilinear chord of the length ρ_{ik}, is

$$M_{ik} = \begin{pmatrix} -\frac{\sin \alpha_i}{\sin \alpha_k} + \frac{\rho_{ik}}{R_i \sin \alpha_k} & -\frac{\rho_{ik}}{\sin \alpha_i \sin \alpha_k} \\ \\ -\frac{\rho_{ik}}{R_i R_k} + \frac{\sin \alpha_k}{R_i} + \frac{\sin \alpha_i}{R_k} & -\frac{\sin \alpha_k}{\sin \alpha_i} + \frac{\rho_{ik}}{R_k \sin \alpha_i} \end{pmatrix} \tag{9}$$

III. CLASSICAL DYNAMICS OF THE TRUNCATED ELLIPTICAL BILLIARD

A. Billiards with $\delta < \sqrt{1 - \gamma^2}$

This subfamily of truncated elliptical billiards has elongated elliptical arcs. In Fig. 3(a-d) we show Poincaré sections for $\delta = 0.19$ and different values of $\gamma < 0.982$. Similar results for $\delta = 0.60$ and $\gamma \leq 0.80$ are shown in Fig. 4(a-d). These pictures reveal a highly chaotic behavior. There are no elliptic islands, however, flights of points typical for neutral orbits can be discerned. This is remarkably different from the corresponding results for the elliptical stadium billiards[33], where in the same parameter region there were many fixed points and elliptic islands due to stable pantographic and other orbits.

1. The bow-tie orbit

We investigate the existence and stability of the bow-tie orbit (the lowest pantographic orbit), shown in Fig. 5(a). This orbit exists if the coordinates x and y of the impact point

FIG. 4: Poincaré plots for $\delta = 0.60$ and various γ.

and the derivative y' of the boundary at this point satisfy the equation (7), which now reads

$$2yy' + x(1 - y'^2) = 0 \tag{10}$$

giving as solution the coordinates of the point of impact

$$x = \sqrt{\frac{1 - \delta^2 - 2\gamma^2}{1 - \delta^2 - \gamma^2}} \tag{11}$$

and

$$y = \frac{\gamma^2}{\sqrt{(1 - \delta^2)(1 - \delta^2 - \gamma^2)}} \tag{12}$$

The condition $\delta < x < 1$ that this point should lie on the elliptical part of the boundary leads to the requirement

$$\delta < \sqrt{\frac{2 - \gamma^2 - \sqrt{\gamma^4 + 4\gamma^2}}{2}}; \quad \gamma < \frac{1 - \delta^2}{\sqrt{2 - \delta^2}} \tag{13}$$

This limit is shown in Fig. 2 and is denoted with the letter a. If we denote the points with positive x by 1 and the points on the negative side by -1, the deviation matrix can be calculated as

$$M = (M_{11}M_{1-1})^2 \tag{14}$$

The corresponding angle α needed in the matrix (9) is given by

$$\sin \alpha = \sqrt{\frac{1 - \delta^2}{2(1 - \delta^2 - \gamma^2)}} \tag{15}$$

The chords are

$$\rho \equiv \rho_{1,1} = 2y = \frac{2\gamma^2}{\sqrt{(1 - \delta^2)(1 - \delta^2 - \gamma^2)}} \tag{16}$$

and

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FIG. 5: Typical lowest order periodic orbits in TEB billiards: (a) bow-tie orbit (neutral); (b) rectangular orbit (neutral); (c) horizontal two-bounce orbit (stable); (d) tilted two-bounce orbit (neutral); (e) diamond orbit (stable); (f) multidiamond orbit with $n = 2$ (stable); (g) hour-glass orbit (neutral); (h) 8-shaped orbit (stable).

$$\rho' \equiv \rho_{1,-1} = 2\sqrt{x^2 + y^2} = 2\sqrt{\frac{1 - \delta^2 - \gamma^2}{1 - \delta^2}} \tag{17}$$

The curvature radius at the impact point is obtained by substituting (11) into (4) and reads

$$R = \frac{2\gamma^2\sqrt{2}}{1 - \delta^2} \tag{18}$$

If we define

$$\Phi = \frac{\rho}{R\sin\alpha}\frac{\rho'}{R\sin\alpha} - \left(\frac{\rho}{R\sin\alpha} + \frac{\rho'}{R\sin\alpha}\right) \tag{19}$$

the trace of the deviation matrix is

$$\mathrm{Tr}M = 2\left[2(2\Phi + 1)^2 - 1\right] \tag{20}$$

The left-hand side of the stability condition (8) is valid automatically, but the right-hand side is fulfilled only if

$$-1 < \Phi < 0 \tag{21}$$

By substituting (15), (16), (17) and (18) into (19), one obtains $\Phi = -1$ for all allowed cases. The conclusion is that the bow-tie orbit is neutral for all parameter values satisfying the existence condition.

2. The rectangular orbit

Further we investigate properties of the rectangular orbit shown in Fig. 5(b). According to (7), this orbit exists if the derivative on the boundary is $y' = -1$. Corresponding solutions for the impact point are

$$x = \sqrt{\frac{1 - \delta^2}{1 - \delta^2 + \gamma^2}} \tag{22}$$

and

$$y = \frac{\gamma^2}{\sqrt{(1 - \delta^2)(1 - \delta^2 + \gamma^2)}} \tag{23}$$

The condition $\delta < x < 1$ leads to the requirement

$$\delta < \frac{\sqrt{\gamma^2 + 4} - \gamma}{2}; \quad \gamma < \frac{1 - \delta^2}{\delta} \tag{24}$$

This limit is shown in Fig. 2 denoted by letter f. Stability is calculated with equation (9) and the matrix (14), where the angle α is given by $\sin \alpha = 1/\sqrt{2}$. The chords are $\rho \equiv \rho_{1,1} = 2x$ and $\rho' \equiv \rho_{1,-1} = 2y$ and the curvature radius is

$$R = \frac{2\sqrt{2}\gamma^2}{1 - \delta^2} \left[\frac{1 - \delta^2}{1 - \delta^2 + \gamma^2} \right]^{3/2} \tag{25}$$

Again, the trace is given by (20), and for this case one obtains $\Phi = 0$. The conclusion is that also this orbit is neutral for all shapes, both flattened and elongated, allowed by (24).

The elongated truncated elliptical billiards were discussed in [29]. Their boundary shapes were described by means of two parameters h and a, related to our parameters δ and γ as follows:

$$h = \sqrt{1 - \delta^2}; \quad a = \frac{\sqrt{1 - \delta^2}}{\gamma} \tag{26}$$

In [29] the billiards with $a > 1$ and $h < 1$ have been analysed and the hyperbolic behavior has been identified in the region $h < \min(1/a, 1/\sqrt{2})$. In our parameters, this corresponds to the quasi-triangular region in the parameter space, denoted by A and B in Fig. 2, delimited by curves $\delta = \sqrt{1 - \gamma}$ (denoted in Fig. 2 by letter e) and $\delta = \sqrt{1 - \gamma^2}$ and by the straight line $\delta = 1/\sqrt{2}$.

The region $h < 1/\sqrt{1 + a^2}$ is rigorously proved to be ergodic[29]. Written with our parameters, it obeys the conditions

$$\frac{\sqrt{\gamma^4 + 4} - \gamma}{2} < \delta < \sqrt{1 - \gamma^2}; \tag{27}$$

$$\frac{1 - \delta^2}{\delta} < \gamma < \sqrt{1 - \delta^2} \tag{28}$$

The corresponding part of the parameter space in Fig. 2 is the one denoted by A. The comparison with our results shows that the limit (27) or (28) is identical to the limit (24) in the parameter space, where the parabolic rectangular orbits emerge.

B. Billiards with $\delta > \sqrt{1 - \gamma^2}$

This part of the parameter space, with flattened elliptical arcs, had not been investigated previously. In Fig. 3(e-h) we show Poincaré sections for $\delta = 0.19$ and different values of $\gamma > 0.98$. In this parameter region dynamics is following the KAM scenario. Similar behavior is noticed for values $\delta = 0.60$ and $\gamma > 0.89$ (Fig. 4(e-h)). Elliptic islands corresponding to the horizontal two-bounce and some other orbits are visible, similarly to the corresponding results for the elliptical stadium billiards[33]. We investigate the existence and stability criteria for these orbits.

1. Horizontal diametral two-bounce orbits

The horizontal two-bounce orbit (Fig. 5(c)) obviously exists for all combinations of δ and γ, but according to [13, 33] the stability condition

$$\frac{\rho}{2R} < 1 \tag{29}$$

takes the form

$$\delta > \sqrt{1 - \gamma^2}; \quad \gamma > \sqrt{1 - \delta^2} \tag{30}$$

so that bifurcations giving birth to stable diametral orbits appear at the values $\delta = \sqrt{1 - \gamma^2}$, corresponding to circular arcs. In the Poincaré diagrams this orbit and the surrounding quasiperiodic orbits are visible as two large bands near $v_x = \pm 1$.

2. Tilted diametral two-bounce orbits

According to (7) a tilted two-bounce orbit (Fig. 5(d)) exists at the point $T(x, y)$ on the billiard boundary with derivative y' if

$$yy' + x = 0 \tag{31}$$

This is realized for any $\delta < x < 1$ provided that

$$\gamma^2 + \delta^2 = 1 \tag{32}$$

thus only for the truncated circle. Since in this case the chord in (29) is $\rho = 2$ and the radius is $R = 1$, these orbits are neutral.

3. Diamond orbit

The diamond orbit of period four, shown in Fig. 5(e), exists for any parameter choice. It has two bouncing points at the ends of the horizontal semiaxis, and the other two on the flat parts on the boundary. To assess its stability, one should calculate the stability matrix $M = (M_{01} M_{10})^2$. The angles contained in the matrix are given as

$$\sin \alpha_0 = \frac{\gamma}{\rho}; \quad \sin \alpha_1 = \frac{1}{\rho} \tag{33}$$

where

$$\rho = \sqrt{1 + \gamma^2} \tag{34}$$

The curvature radius at $x = 1$ is given by (6). This leads to the trace

$$\mathrm{Tr} M = 2 \left[2 \left(\frac{2\rho^2}{R} - 1 \right)^2 - 1 \right] \tag{35}$$

and to the condition $\rho^2 < R$ or $1 + \gamma^2 < \gamma^2/(1 - \delta^2)$, thus the stable diamond orbit appears when

$$\delta > \frac{1}{\sqrt{1+\gamma^2}}; \quad \gamma > \sqrt{\frac{1}{\delta^2} - 1} \qquad (36)$$

This limit is shown in Fig. 2 as the line denoted by letter c.

4. Multidiamond orbits

The multidiamond orbit of order n is the orbit of period $2 + 2n$, which has two bouncing points at the ends of the horizontal axis and $2n$ bouncing points on the flat parts of the boundary (Fig. 4 (f)). Such orbit exists if

$$\delta > 1 - \frac{1}{n} \qquad (37)$$

As explained for a similar case in [33], the chord ρ in (35) should be replaced by

$$L = n\rho_n \qquad (38)$$

where, for the truncated elliptical billiard,

$$\rho_n = \sqrt{\frac{1}{n^2} + \gamma^2} \qquad (39)$$

The trace of the stability matrix is then

$$\mathrm{Tr}M = 2\left[2\left(\frac{2\rho_n^2 n^2}{R} - 1\right)^2 - 1\right] \qquad (40)$$

with R given by (6). The resulting condition for the stability of the multidiamond orbit is

$$\delta > \sqrt{1 - \frac{\gamma^2}{1 + \gamma^2 n^2}}; \quad \gamma > \sqrt{\frac{1 - \delta^2}{1 - n^2(1 - \delta)}} \qquad (41)$$

The limiting curves (41) are plotted in Fig. 2. The line with $n = 2$ is denoted by letter d, and above it there are several lines for $n > 2$. For $\gamma \to \infty$ the minimal values of δ above which the multidiamond orbits appear are

FIG. 6: Poincaré plots for a set of different values $\delta = \gamma$, showing the appearance of successive multidiamond orbits of higher order. Two symmetrical triangle-shaped islands for $\delta = \gamma = 0.90$ originate from the "8-shaped" orbit.

$$\lim_{\gamma \to \infty} \delta = \sqrt{1 - \frac{1}{n^2}} \tag{42}$$

The emergence of multidiamond orbits can be followed by observing the Poincaré sections for a set of shapes with $\delta = \gamma$ (Fig. 6). The values of this parameter for which an orbit of new n appears are given as intersections of the straight line $\delta = \gamma$ with curves (41), and obey the equation

$$n^2 \delta^4 - (n^2 - 2)\delta^2 - 1 = 0 \tag{43}$$

For the diamond orbit ($n = 1$) this equation reads

$$\delta^4 + \delta^2 - 1 = 0 \tag{44}$$

and the orbit appears for

$$\delta = \gamma = \sqrt{(\sqrt{5} - 1)/2} = 0.78615.$$

For the same type of boundary the stable two-bounce orbit appeared at

$$\delta = \gamma = 1/\sqrt{2} = 0.707107.$$

5. *The hour-glass orbit*

The hour-glass orbit (Fig. 5(g)) looks like the bow-tie orbit rotated by $\pi/2$. It exists if the coordinates x and y of the impact point and the derivative y' of the boundary at this point satisfy the equation

$$2xy' + yy'^2 - y = 0 \tag{45}$$

giving as solution the coordinates of the impact point

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$$x = \sqrt{\frac{1 - \delta^2}{\delta^2 + \gamma^2 - 1}} \tag{46}$$

and

$$y = \gamma \sqrt{\frac{2\delta^2 - 2 + \gamma^2}{(1 - \delta^2)(\delta^2 + \gamma^2 - 1)}} \tag{47}$$

The condition $\delta < x < 1$ that this point should lie on the elliptical part of the boundary leads to the requirement

$$\sqrt{1 - \frac{\gamma^2}{2}} < \delta < \sqrt{\frac{-\gamma^2 + \sqrt{\gamma^2 + 4}}{2}}$$

$$\sqrt{2(1 - \delta^2)} < \gamma < \frac{\sqrt{1 - \delta^4}}{2} \tag{48}$$

These limits define the region shown in Fig. 2 denoted by letter b.

If we denote the points with positive y by 1 and the points on the negative side by -1, the deviation matrix can be calculated from (14). The angle α needed in the calculation is given as

$$\sin \alpha = \sqrt{\frac{\gamma^2}{2(\delta^2 + \gamma^2 - 1)}} \tag{49}$$

The curvature radius at this point is given as

$$R = \sqrt{\frac{8(1 - \delta^2)}{\gamma^2}} \tag{50}$$

The chords are

$$\rho \equiv \rho_{1,1} = 2x = 2\sqrt{\frac{1 - \delta^2}{\delta^2 + \gamma^2 - 1}} \tag{51}$$

and

$$\rho' \equiv \rho_{1,-1} = 2\sqrt{x^2 + y^2} = 2\sqrt{\frac{\delta^2 + \gamma^2 - 1}{1 - \delta^2}} \tag{52}$$

If we define Φ as in (19), the trace of the deviation matrix is again given by (20) and the orbit is stable if $-1 < \Phi < 0$.

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FIG. 7: Diagram showing the degree of chaoticity q_{class} of the truncated elliptical billiards (TEB), in dependence on the shape parameters, with $0 < \gamma < 2.2$ and $0 \leq \delta \leq 1$. Black points denote shapes with $q_{\text{class}} = 1.00$, red with $0.90 \leq q_{\text{class}} < 1.00$, green with $0.80 \leq q_{\text{class}} < 0.90$, yellow with $0.70 \leq q_{\text{class}} < 0.80$, blue with $0.60 \leq q_{\text{class}} < 0.70$ and grey with $0.00 \leq q_{\text{class}} < 0.60$.

When we substitute the calculated values of R, ρ, ρ' and $\sin \alpha$ into (19), we obtain $\Phi = -1$ for all allowed shapes and conclude that the hour-glass orbit is neutral. This means that in the truncated elliptical billiards (TEB) there is no stable hour-glass orbit, at variance with the elliptical stadium billiard (ESB), where such an orbit having interesting properties was stable in a large fraction of the parameter space[33]. Besides the diamond and multidiamond orbits, in Fig. 6 one discerns the presence of another, "8-shaped", stable orbit, shown in Fig. 5(h).

IV. THE BOX-COUNTING NUMERICAL ANALYSIS OF THE DEGREE OF CHAOTICITY IN THE FULL PARAMETER SPACE

In this section we return to the question of limits within which the truncated elliptical billiard is fully chaotic. Here we test these limits numerically, with the help of the box-counting method[14, 33, 38]. We calculate the Poincaré sections for a chosen pair of shape parameters, starting with n_1 randomly chosen sets of initial conditions and iterating each orbit for n_2 intersections with the x-axis, thus obtaining $n_1 \times n_2$ points in the Poincaré diagram. Then we divide the first quadrant of the phase plane into a grid of $n \times n$ squares (boxes), count the number of boxes which have points in them and calculate the ratio of this number to the total number of boxes. The obtained ratio is denoted by q_{class}. In this way also certain points belonging to invariant curves within the regular islands are included. But since our main aim is to examine the onset of full chaos, this method gives satisfactory results, providing that the appropriate values of n_1, n_2 and n are used. Detailed testing has shown that reliable results are obtained for values $n_1 = 100$, $n_2 = 5000$ and $n = 100$ used in our present calculation[39].

In Fig. 7 we plot in the $\delta - \gamma$ plane the points representing the pairs of shape parameters. Points are plotted in different colors, depending on the corresponding value of q_{class}. The full chaos, corresponding to $q_{\text{class}} = 1.00$, is depicted by black points. Colored points denote

FIG. 8: Diagram showing the degree of chaoticity q_{class} of the elliptical stadium billiards (ESB), in dependence on the shape parameters, with $0 < \gamma < 2.2$ and $0 \leq \delta \leq 1$. Black points denote shapes with $q_{class} = 1.00$, red with $0.90 \leq q_{class} < 1.00$, green with $0.80 \leq q_{class} < 0.90$, yellow with $0.70 \leq q_{class} < 0.80$, blue with $0.60 \leq q_{class} < 0.70$ and grey with $0.00 \leq q_{class} < 0.60$.

FIG. 9: Construction and parameters of the generalized elliptical truncated stadium-like billiards (GTESB)

shapes within intervals between 0 and 0.99. This diagram confirms that for the truncated elliptical billiards (TEB), in the region below the onset of the stable two-bounce horizontal orbit, dynamics is practically completely chaotic. This is in strong contrast with the behavior of the elliptical stadium billiard for which the similar diagram is shown in Fig. 8. For the ESB billiards the region of chaos was strictly bounded also from the lower side and determined by emergence of stable pantographic orbits.

To examine the possible mechanism for this difference, we assume that the TEB and the ESB billiards are two extreme cases and search for a possible transition between them.

V.   GENERALIZED TRUNCATED STADIUM-LIKE ELLIPTICAL BILLIARDS

In this section we propose a new large class of stadium-like billiards which we call generalized truncated elliptical stadium-like billiards (GTESB). Such a billiard depends on three shape parameters δ, γ and κ. The allowed values of the shape parameters are

$$0 \leq \delta \leq 1; \ 0 < \gamma < \infty; \ -1 \leq \kappa \leq 1 \tag{53}$$

For the limiting values of κ we obtain the two billiard families considered before: for $\kappa = -1$ GTESB reduces to the elliptical stadium billiard (ESB), and for $\kappa = 1$ GTESB becomes the truncated elliptical billiard (TEB).

The new GTESB billiard boundary is obtained by adding elliptical arcs symmetrically at the two opposite ends of a rectangle with sides 2δ and 2γ. Elliptical arcs are cut out from the two identical but generally detached ellipses by two horizontal straight lines at $y = \pm\gamma$

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FIG. 10: Poincaré plots for the generalized truncated elliptical stadium-like billiards (GTESB), for $\delta = 0.2$ and $\gamma = 0.6$ with different values of κ: (a) $\kappa = -1.0$; (b) $\kappa = -0.6$; (c) $\kappa = -0.3$; (d) $\kappa = 0$;(e) $\kappa = 0.3$; (f) $\kappa = 0.6$; (g) $\kappa = 0.90$; (h) $\kappa = 1.0$.

(Fig. 9). The two ellipses have centers at the points

$$X_\pm = \pm\delta\left(\frac{1-\kappa}{2}\right) \tag{54}$$

The distance between the two centers is $D = \delta(1 - \kappa)$. The horizontal and vertical semiaxis are given, respectively, as

$$A_\mathrm{x} = 1 - \delta\left(\frac{1-\kappa}{2}\right) \tag{55}$$

and

$$A_\mathrm{y} = \frac{\gamma}{\sqrt{(1-\delta)(1+\kappa\delta)}}\left[1 - \delta\left(\frac{1-\kappa}{2}\right)\right] \tag{56}$$

and the equation of the two ellipses reads

$$\left(\frac{x - X_\pm}{A_\mathrm{x}}\right)^2 + \left(\frac{y}{A_\mathrm{y}}\right)^2 = 1 \tag{57}$$

The horizontal diameter of the billiard is 2. For $\kappa = -1$ and $\gamma = 1 - \delta$ the GTESB becomes the Bunimovich stadium billiard.

In Fig. 10 the Poincaré sections are shown for $\delta = 0.2$ and $\gamma = 0.6$ with κ assuming different values between -1 and 1. The four islands typical for the bow-tie orbit are present for all κ except for $\kappa = 1$ (TEB), where this orbit becomes neutral and the island reduces to a caracteristical flight of points. The limits separating chaotic from mixed behavior are determined by the onset of the stable horizontal 2-bounce orbit and are given by (29). Since the curvature radius at $|x| = 1$ is

$$R_1 = \frac{\gamma^2[2 - \delta(1 - \kappa)]}{2(1 - \delta)(1 + \delta\kappa)} \tag{58}$$

the upper limit of chaos is determined by the condition

FIG. 11: Dependence of q_{class} on γ for a set of shapes with $\delta = \gamma$ and the choice of κ as in Fig.10.

$$\gamma > \sqrt{\frac{2(1-\delta)(1+\kappa\delta)}{2-\delta(1-\kappa)}} \tag{59}$$

In Fig. 11 the chaotic fraction q_{class} is shown for the special case $\delta = \gamma$ for different κ, in dependence on γ. It is noticed that in the case $\kappa = -1$ (ESB) there is a narrow, strictly limited region of full chaos, outside of which the values of q_{class} are low. For $\kappa = 1$ (TEB) the fully chaotic region is much larger and extends practically over all values δ and γ below the chaotic limit. Between these two limits, the regions of full chaos ($q_{\text{class}} = 1$) are shorter and limited, but there are many shapes with chaotic parameter close to 1 (between 0.90 and 0.99). This corresponds to a selection of narrow islands in the Poincaré plots, as seen in Fig. 10.

VI. DISCUSSION AND CONCLUSIONS

In conclusion, our investigation of the elliptical stadium-like billiards has revealed a rich variety of integrable, mixed and chaotic behavior, which is connected with the character of the two elliptical arcs and with their mutual position. This strong dependence on parameters δ and γ is confirmed for the truncated elliptical billiards, but is even more enhanced when a third shape parameter κ is added. Analysis shows, however, that among all considered shapes the truncated elliptical billiard (TEB), created by cutting a single ellipse with two parallel straight lines, has exceptional properties, notably that it is chaotic practically in the whole region of elongated elliptical arcs. Notable is the presence of many neutral orbits in this region, consistent with the fact that these orbits actually can be identified as orbits in an ellipse. For the flattened arcs, the stable islands due to the two-bounce horizontal orbit and to the diamond and multidiamond orbits occupy an important part of the phase plane.

Our investigations can be useful for the experimental application of billiards in the laser technology, where properties and directional intensities of the optical microresonators depend strongly on the boundary shape. They can also be applied in designing the semiconducting optical devices and in the technology of microwave and acoustic resonant cavities. With this purpose in mind, we propose further analysis of the stadium-like billiards with elliptical arcs and an extension of the present investigation to different types of open billiards.

VII. ACKNOWLEDGMENTS

[1] G.M. Zaslavsky, Physics of Chaos in Hamiltonian systems, Imperial College Press, London 2007

[2] Ya. G. Sinai, Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards, Russ. Math. Surv. **27**, 137 (1970)

[3] L.A. Bunimovich, Funct. Anal. Appl. **8**, 254 (1974)

[4] L.A. Bunimovich, Commun. Math. Phys. **65**, 295 (1979)

[5] N. Chernov and R. Markarian, Chaotic Billiards, American Mathematical Society, 2006

[6] G. DelMagno and R. Markarian, Commun. Math. Phys. **233**, 211 (2003)

[7] V.J. Donnay, Commun. Math. Phys. **141**, 225 (1991)

[8] R. Markarian, Nonlinearity **6**, 819 (1993)

[9] M. Wojtkowski, Commun. Math. Phys. **105**, 391 (1986)

[10] G. Benetin and J.-M. Strelzyn, Phys. Rev. A **17**, 773 (1978)

[11] H. R. Dullin and A. Bäcker, Nonlinearity **14**, 1673 (2001)

[12] M.V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984)

[13] M. Berry, Eur. J. Phys. **2**, 91 (1981)

[14] H. R. Dullin, P. H. Richter and A. Wittek, Chaos **6**, 43 (1996)

[15] E.J. Heller and S. Tomsovic, Phys. Today **46**(7), 38 (1993)

[16] H. Makino, T. Harayama and Y. Aizawa, Phys. Rev. E **59**, 4026 (1999)

[17] G. Casati and T. Prosen, Phys. Rev. Lett. **83**, 4729 (1999)

[18] S. Ree and L. E. Reichl, Phys. Rev. E **60**, 1607 (1999)

[19] M. Robnik, J. Phys. A **16**, 3971 (1983)

[20] C. Gmachl, F. Capasso, J.U. Nöckels, A.D. Stone, D.L. Sivco, and A.Y. Cho, Science **280**, 1556 (1998)

[21] T. Harayama, P. Davis and K.S. Ikeda, Phys. Rev. Lett. **90**, 073903 (2003)

[22] M. Hentschel and K. Richter, Phys. Rev. E **66**, 056297 (2002)

[23] M. Lebental, N. Djellali, C. Arnaud, J.-S. Lauret, J. Zyss, R. Dubertrand, C. Schmit and L. Bogomolny, Phys. Rev. A **76**, 023830 (2007)

[24] H.E. Tureci, H.G.L. Schwefel, A. Douglas Stone and E.E. Narimanov, Optics Express **10**, 752 (2002)

[25] T. Tanaka, M. Hentschel, T. Fukushima and T. Harayama, Phys. Rev. Lett. **98**, 033902 (2007)

[26] E. Canale, R. Markarian, S. Oliffson Kamphorst and S. Pinto de Carvalho, Physica D **115**, 189 (1998)

[27] R. Markarian, S. Oliffson Kamphorst and S. Pinto de Carvalho, Commun. Math. Phys. **174**, 661 (1996)

[28] S. Oliffson Kamphorst and S. Pinto de Carvalho, Discr. and Cont. Dynam. Syst. **7**, 663 (2001)

[29] G. Del Magno, Nonlinearity **14**, 1761 (2001)

[30] V. Lopac, I. Mrkonjić and D. Radić, Phys. Rev. E **66**, 035202 (2002)

[31] V. Lopac, I. Mrkonjić and D. Radić, Phys. Rev. E **64**, 016214 (2001)

[32] V. Lopac, I. Mrkonjić and D. Radić, Phys. Rev. E **59**, 303 (1999)

[33] V. Lopac, I. Mrkonjić, N. Pavin and D. Radić, Physica D **217**, 88 (2006)

[34] V. Lopac, I. Movre, I. Mrkonjić and D. Radić, Prog. Theor. Phys. Suppl. **150**, 371 (2003)

[35] E.G. Altmann, A.E. Motter and H. Kantz, Chaos **15**, 033105 (2005)

[36] L.A. Bunimovich, Chaos **11**, 802 (2001)

[37] S. Lansel, Mason A. Porter and L.A. Bunimovich, Chaos **16**, 013129 (2006)

[38] T. Prosen and M. Robnik, J. Phys. A: Math. Gen. **31**, L345 (1998)

[39] A. Šimić, B. Sc. Thesis, Faculty of Science, University of Zagreb (2009)

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